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COMMENT

A renormalisation group approach to the viscous-finger fractal at finite viscosity ratio

Takashi Nagatani

College of Engineering, Shizuoka University, Hamamatsu 432, Japan

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Abstract. A real-space renormalisation group method is applied to viscous fingering to analyse the fractal nature at a finite viscosity ratio. The renormalisation group transformation is obtained for the permeability of the surface layer which plays the role of coupling constant in the renormalisation group of the phase transition. A stable fixed point is found as a function of the viscosity ratio. The growth probabilities on the perimeter bonds of the injected fluid are evaluated at the fixed point. The multifractal scaling properties are found to be described by the generalised dimensions and the α - f spectra. It is shown that at a finite viscosity ratio the displaced area is compact with a surface fractal dimension between 1 and the DLA result with increasing viscosity ratio. This result is consistent with that of King's numerical simulation.

Recently, there has been increasing interest in the problem of geometrical structure in kinetic growth models [1-11]. The structure of the aggregates strongly depends on the dynamics of the growth process. An analogy [12] has been suggested between the diffusion-limited aggregation (DLA) model of Witten and Sander [1] and viscous fingering with a zero-viscosity displacing fluid. Since then many attempts have been made to modify DLA to include surface tension effects [13, 14]. However, fewer attempts to allow for a finite viscosity ratio [15, 16] have been made. The question of the nature of the fingered patterns at a finite viscosity ratio has been left unanswered. Very recently, King [17] presented a numerical simulation for the fingered patterns at a finite viscosity by using a network model of the porous medium. He found that at a finite viscosity ratio the displaced area is compact with a surface fractal dimension between 1 and the DLA result with increasing viscosity ratio.

In this comment, we extend the real-space renormalisation group method for the DLA [18-20] to analyse the fractal nature of the viscous-finger fractal at a finite viscosity ratio. We derive the surface fractal dimensions as a function of the viscosity ratio. The multifractal structure is studied with varying viscosity ratio. We suggest an analogy between the viscous finger at a finite viscosity ratio and the electro-deposited aggregate at a finite conductivity ratio.

When a fluid is forced into a porous medium to displace another more viscous fluid the interface between the two fluids develops a highly complex fingered pattern. The growth of the injected fluid occurs on the perimeter of the finger. In our model the growth probability \tilde{p}_i at the growing perimeter bond i is given by $\tilde{p}_i \sim \text{grad } P_i$ where \tilde{p}_i is the probability that bond i becomes part of the finger and $\text{grad } P_i$ is the local pressure gradient at the growth bond. The basic equation governing the viscous fingering is given by

$$k\nabla^2 P = 0 \tag{1}$$

where P , k and ∇^2 represent the pressure, permeability and Laplacian, respectively. The permeability is proportional to the inverse of the viscosity. The viscosity ratio M is defined as $\eta_D/\eta_I (= k_I/k_D)$ where η_I is the viscosity of the injected fluid and η_D is the viscosity of the displaced fluid. We consider the discrete version of equation (1). We merely solve a Laplace equation for a finger cluster inside an infinite more viscous fluid network. Here we assume that the dynamics of the formation of a viscous finger is determined by a 'quasi-stationary' pressure field, similar to DLA [21]. The process of the formation of a finger is a purely dissipative system without convection in a non-equilibrium steady state. The process is characterised by the theorem of minimum entropy production or minimum dissipation energy. We consider the dissipation energy in contrast to the free energy (or Hamiltonian). The dissipation energy is given by

$$H_d = \sum_i k_{i,i} (\nabla P_i)^2 + \sum_j k_{D,j} (\nabla P_j)^2 = k_t (\nabla P_t)^2 \tag{2}$$

where the first term represents the summation of local dissipation energy of each bond over the total bonds within the system, and the second term indicates the total dissipation energy in the representation of the total permeability k_t and the total pressure drop ∇P_t of the system. Thus the dissipation energy form (2) describes the interaction between neighbouring potentials by this simple gradient term ∇P_t . The total flow through the interface of the finger is given by $J_t = \sum_i k_i \nabla P_i$ where the sum indicates the summation over all the bonds on the growing perimeter. We can write the dissipation energy in terms of the surface permeability:

$$H_d = (\nabla P_t) \left(\sum_{i_s} k_{i_s} \nabla P_{i_s} \right) \tag{3}$$

where k_{i_s} indicates the permeability of the bonds on the perimeter of the finger. By using a decimation method, one may write the dissipation energy in terms of the coarse-grained variables k'_{i_s} , $\nabla P'_{i_s}$ on the new lattice with lattice spacing b (b is a scale factor):

$$H_d = (\nabla P_t) \left(\sum_{i'_s} k'_{i'_s} \nabla P'_{i'_s} \right). \tag{4}$$

Thus the short-range interaction between neighbouring pressures caused by the gradient term is renormalised. The surface permeability k_G is transformed to k'_G . This surface permeability plays a role of the so-called coupling constant in the Hamiltonian of the phase transition. We shall derive the renormalisation group equation for the surface permeability: $k'_G = R(k_G)$. We must distinguish between three types of bonds on the lattice before a renormalisation procedure: (a) interior bonds which are occupied by injected fluid and construct the finger; (b) growth bonds which are on the interface of the finger, occupied by the displaced fluid and can be successively grown; and (c) exterior bonds which are occupied by the displaced fluid except for the growth bonds. Figure 1(a) shows a part of the finger near the interface where the interior, growth and exterior bonds are respectively indicated by the bold, wavy and full lines. Cover all the space of the square lattice by cells of edge b (scale factor), each containing $2b^2$ bonds; an example for $b = 2$ is shown in figure 2. After a renormalisation transformation these cells play the role of 'renormalised' bonds. The renormalised bonds are then classified into three types of bonds, similarly to the bonds before the renormalisation. The renormalisation procedure is indicated by figure 2. If the cell is spanned with the bonds occupied by the injected fluid then the renormalised bond is considered to be

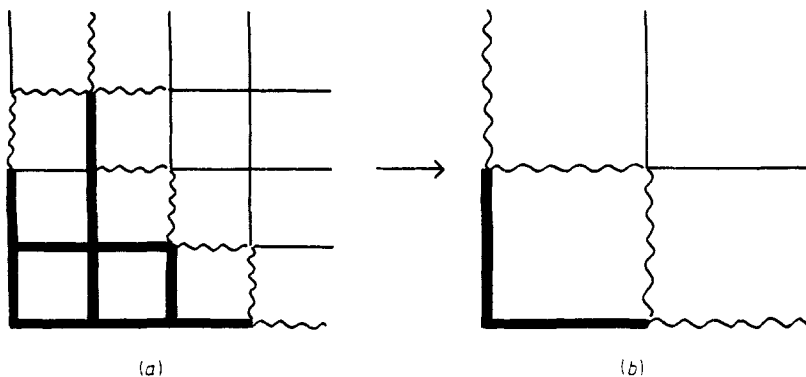


Figure 1. An example of the renormalisation of a part of the surface layer of a finger. Bonds constructing the finger are indicated by bold lines and bonds on the perimeter of the finger by wavy lines. The lattice on the left-hand side is renormalised to that on the right-hand side, according to the rule of renormalisation.

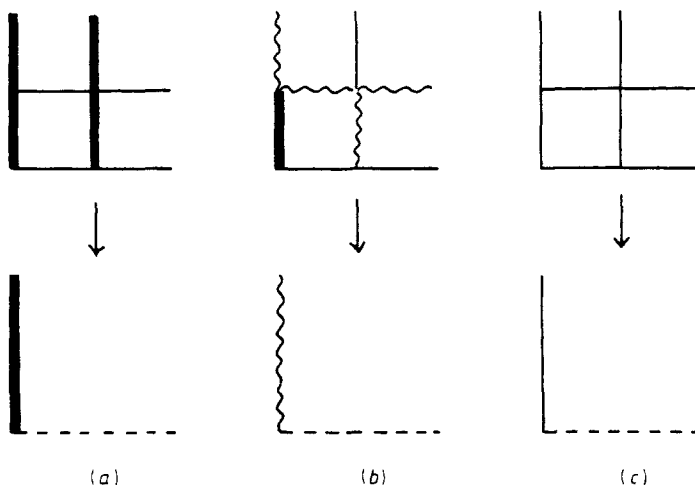


Figure 2. Illustration of the renormalisation of a $b=2$ cell for viscous fingering. The renormalisation procedure in the vertical direction is shown. There are three types of bonds: interior bonds indicated by bold lines, growth bonds by wavy lines and exterior bonds by full lines. Examples of the distinct configurations are shown in (a), (b) and (c), which are renormalised as interior, growth and exterior bonds, respectively.

interior (figure 2(a)). If the cell is not spanned with the injected fluid and is nearest neighbour to the finger, then the cell is renormalised as the growth bond on the interface (figure 2(b)). When the cell is constructed by the displaced fluid only and is not nearest neighbour with the finger cluster, the cell is renormalised as the displaced fluid (figure 2(c)). Figure 1 shows an example of the renormalisation of a part of the finger near the interface. The lattice on the left-hand side is renormalised to that on the right-hand side, according to the rules of renormalisation. We consider the permeabilities of the three types of the bonds. After many repeated renormalisations, each permeability of the three types of bonds must approach each finite value because the total permeability k_t of the system has a finite value, except for zero. The permeability of the cell to

be renormalised as the exterior bond does not change before and after the renormalisation because all the bonds within the cell are occupied by displaced fluid. The condition that the permeability of the cell to be renormalised as the interior bond has a finite value, except zero, is satisfied by constructing the cell within which all the bonds are occupied by the injected fluid. Otherwise, the permeability of the interior bond approaches the zero value after many repeated renormalisations and the growth of the finger stops. Hence the interior of the fingers is compact. We consider the permeability of the cell that can be renormalised as the growth bond. We define the permeability of the growth bond as a surface permeability. The non-local nature of the pressure field is taken into account as the permeability of the growth bond. If one considers the renormalisation in the vertical direction, we shall take periodic boundary conditions in the lateral direction. The constant voltage is vertically applied. Figure 3 shows all configurations of the cell that it is possible to renormalise as the growth bond. Let us consider the configurational probability C_α with which a particular configuration α appears. The distinct configurations are labelled by α ($\alpha = 0, 1, 2, 3$) in figure 3. Configuration (1) is constructed by adding an interior bond to configuration (0). In addition, by adding an interior bond to configuration (1), configurations (2) or (3) occur. Furthermore, we introduce another route, in which configuration (3) is directly constructed from configuration (0). The interface grows uniformly with a finite probability because the interior of the finger is compact. We assume that the probability A growing uniformly is proportional to the permeability ratio k_G/k_I where k_G and k_I indicate respectively the permeabilities of the growth bond and interior bond. At the limit of the zero-viscosity displacing fluid, the uniformly growing probability A becomes zero. The configurational probabilities C_α are given by

$$\begin{aligned}
 C_1 &= (1 - A)C_0(p_{0,1} + p_{0,2}) \\
 C_2 &= (1 - A)C_1(p_{1,2} + p_{1,3}) \\
 C_3 &= (1 - A)C_1p_{1,4} + AC_0
 \end{aligned}
 \tag{5}$$

where $A = k_G/k_I$ and $p_{\alpha,i}$ represent the growth probability of the growth bond i within the cell α . The configurational probability C_0 is determined from the normalisation

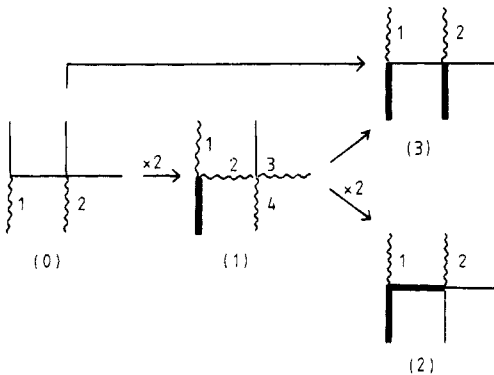


Figure 3. All distinct configurations of the 2×2 cell that it is possible to renormalise as the growth bond. Configuration (1) is constructed by adding an interior bond to configuration (0). In addition, by adding an interior bond to configuration (1), configurations (2) or (3) occur. Also configuration (3) is directly constructed from configuration (0).

condition

$$\sum_{\alpha} C_{\alpha} = C_0 + C_1 + C_2 + C_3 = 1. \tag{6}$$

The growth probability $p_{\alpha,i}$ on the growth bond i within the cell α is proportional to the pressure gradient ∇P_i on the growth bond. Consider the Laplace equation for cells which can be renormalised as the growth bond. The pressure field within a cell is determined by the permeabilities of the interior, exterior and growth bonds and the configuration of the cell. In the configuration labelled by α (see figure 3), the growth probabilities $p_{\alpha,i}$ of growth bonds i are given by

$$\begin{aligned} p_{0,1} &= p_{0,2} = \frac{1}{2} \\ p_{1,2} &= p_{1,3} = [1 - (k_G/k_I)k_G]/[(4 + 3k_G) + (k_G/k_I)(3 + 2k_G)] \\ p_{1,1} &= 1 - 2p_{1,2} - p_{1,4} \\ p_{2,1} &= p_{2,2} = \frac{1}{2} \\ p_{3,1} &= p_{3,2} = \frac{1}{2} \end{aligned} \tag{7}$$

where k_I , k_D and k_G indicate the permeabilities of the interior, exterior and growth bonds, respectively, and by setting $k_D = 1$ the viscosity ratio is given by $M = k_I$. The permeability $k'_{G,\alpha}$ of the cell with configuration α is renormalised as follows:

$$\begin{aligned} k'_{G,0} &= 2k_G/(1 + k_G) \\ k'_{G,1} &= k_G[(4 + 3k_G) + (k_G/k_I)(3 + 2k_G)]/[(1 + 3k_G) + (k_G/k_I)(3 + 5k_G)] \\ k'_{G,2} &= 2k_G/[1 + (k_G/k_I)] \\ k'_{G,3} &= k_{G,2}. \end{aligned} \tag{8}$$

The renormalised permeability k'_G of the growth bond will be assumed to be given by the most probable value

$$k'_G = \exp[C_0 \log(k'_{G,0}) + C_1 \log(k'_{G,1}) + C_2 \log(k'_{G,2}) + C_3 \log(k'_{G,3})]. \tag{9}$$

The relationships (8) and (9) present the renormalisation group equation $k'_G = R(k_G)$. Equations (5)-(9) are simultaneously solved. We find a non-trivial fixed point k_G^* from $k_G^* = R(k_G^*)$ as a function of the viscosity ratio $M (=k_I)$. Figure 4 shows the schematic

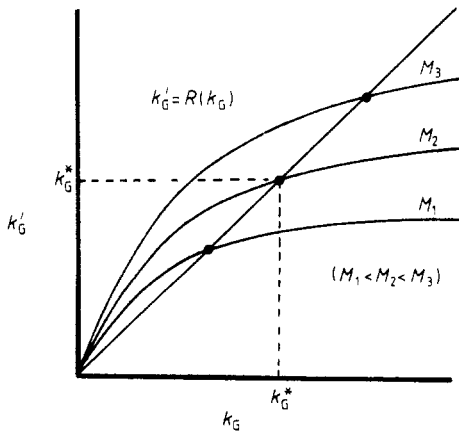


Figure 4. Schematic behaviour of the renormalisation function $k'_G = R(k_G)$ as a parameter of the viscosity ratio M . The RG equation has a stable fixed point between 1 and the DLA result with increasing viscosity ratio.

behaviour of the renormalisation function $k'_G = R(k_G)$ as a parameter of the viscosity ratio $M(=k_i)$. When the viscosity of the injected fluid approaches that of the displaced fluid ($M \rightarrow 1$), the fixed point approaches one ($k_G^*/k_D \rightarrow 1$). In the limit of a zero-viscosity injected fluid, the fixed point becomes that of DLA: $k_G^*/k_D = 2.326$. The RG equation has a fixed point between 1 and the DLA result with increasing viscosity ratio. At the fixed points, the derivatives dR/dk_G have positive values less than one. They are stable fixed points. These results represent stable steady states and are consistent with the entropy production minimum theorem or the dissipation energy minimum theorem [20].

We now consider the growth probability on any growth bond. We define the two growth probabilities: (a) the growth probability $p_i(L)$ over the total perimeter bonds of the finger with the size L and (b) the growth probability $p_{\alpha,i}$ on the growth bond i within the cell α . Before and after renormalisation, the growth probability $\tilde{p}_i(L)$ on any growth bond i is given by

$$\tilde{p}_i(L) = p_{\alpha,i} \tilde{p}_\alpha(L/b) \tag{10}$$

where L represents the size of the system and b is the scale factor. The cell's growth probability $p_{\alpha,i}$ is represented by a function of the permeability of the growth bond as a parameter of the viscosity ratio $M(=k_i)$. After many repeated renormalisations, the growth probability assigned to each growth bond is represented by a random multiplicative process of the cell's growth probabilities evaluated at the fixed point. In the limit of L sufficiently large, an infinite hierarchy of generalised dimensions $D(q)$ is given by

$$D(q) = -(q - 1)^{-1} \log \left\langle \sum_i p_{\alpha,i}^{*q} \right\rangle_c (\log b)^{-1} \tag{11}$$

where $\langle \rangle_c$ indicates the configurational average and $p_{\alpha,i}^*$ represents the cell's growth probability evaluated at the fixed point. The surface fractal dimension d_s is given by $D(0)$. A dependence of the surface fractal dimension on the viscosity ratio M is shown in figure 5. It shows a continuous function of viscosity ratio between one (for unit viscosity ratio) and about 1.526 for the infinite viscosity ratio case analogous to DLA.

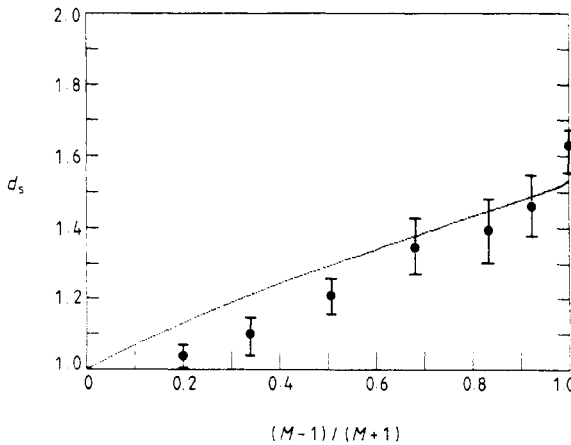


Figure 5. Surface fractal dimension as a function of viscosity ratio. The full circles indicate the numerical result by King [17]

We now show that the cluster-top occupancy probability \tilde{p}_a determines the fractal dimension d_f , following Turkevich's method [22, 23]. Given a (L, M) finger which consists of M bonds and has size L , we obtain

$$\Delta M = (1/\tilde{p}_a)\Delta L = [(1-A)/\tilde{p}_{\max} + A/\tilde{p}_U]\Delta L \tag{12}$$

where \tilde{p}_{\max} and \tilde{p}_U indicate the maximum growth probability and the uniform growth probability, respectively. Since the scalings $(1/\tilde{p}_{\max}) \sim L^{D(\infty)}$ and $(1/\tilde{p}_U) \sim L$ ($0 < D(\infty) < 1$) hold, we obtain the fractal dimension $d_f = 2$ from $M \sim L^2$. Hence the interior

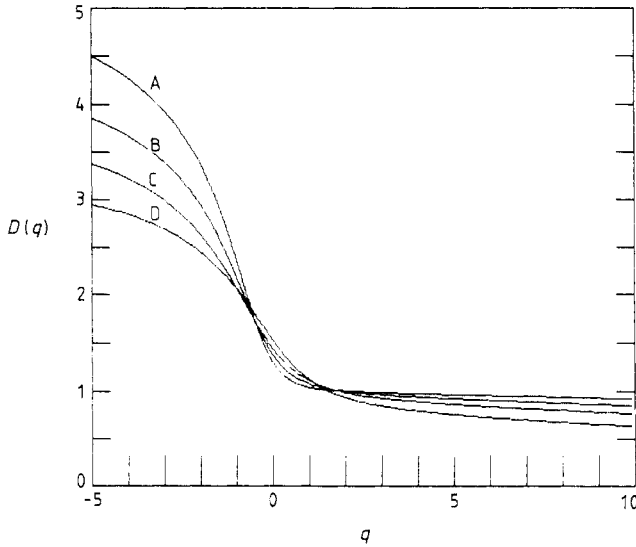


Figure 6. The generalised dimension $D(q)$ of the viscous fingering at a finite viscosity ratio M . The curves A-D indicate the results for $M = 3, 5, 10, \infty$, respectively.

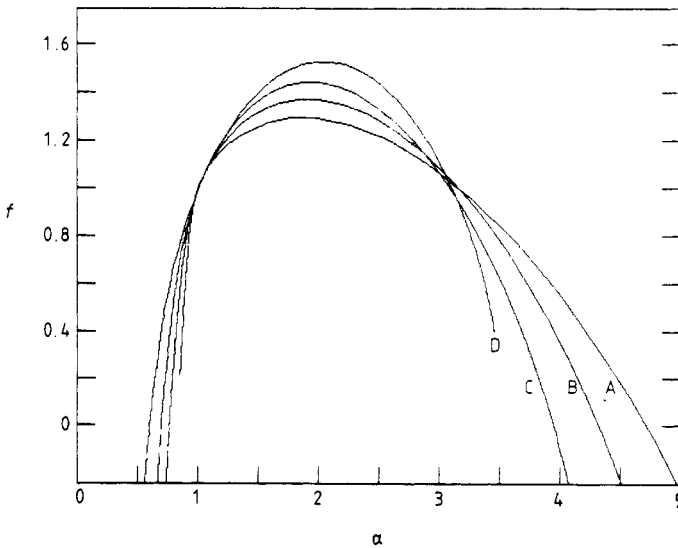


Figure 7. The α - f spectra of the viscous finger fractals at finite viscosity ratios $M = 3$ (A), 5 (B), 10 (C), ∞ (D).

of the finger is compact. We note that $[D(\infty) + 1]$ is not identical with the fractal dimension d_f for viscous fingering at a finite viscosity. The partition of $D(q)$ into a density of singularities $f(q)$ with singularity strength $\alpha(q)$ is introduced:

$$\alpha(q) = d/dq[(q-1)D(q)] \quad f(q) = q\alpha(q) - (q-1)D(q). \quad (13)$$

The multifractal scaling structure is shown in figures 6 and 7: $D(q)$ in figure 6 and the α - f spectra in figure 7. As the viscosity ratio M decreases, the α - f spectrum becomes a more steep curve in the range of positive q . This is analogous to the generalised DLA model (η model) [24–26]. However, in the range of negative q , the α - f spectrum becomes more smooth with decreasing viscosity ratio, in contrast to the η model.

In summary, we have applied the real-space renormalisation group method to the viscous-finger fractal at a finite viscosity ratio. We have analytically found the surface fractal dimension and the α - f spectra depending on the viscosity ratio. We have shown that at a finite viscosity ratio the displaced area is compact with a surface fractal dimension between 1 and the DLA result with increasing viscosity ratio. This result is consistent with that of King's numerical simulation.

References

- [1] Witten T A and Sander L M 1981 *Phys. Rev. Lett.* **47** 1400
- [2] Meakin P 1983 *Phys. Rev. A* **27** 1495
- [3] Family F and Landau D P (ed) 1984 *Kinetics of Aggregation and Gelation* (Amsterdam: North-Holland)
- [4] Stanley H E and Ostrowsky N (ed) 1985 *On Growth and Form* (Dordrecht: Martinus Nijhoff)
- [5] Pietronero L and Tosatti E (ed) 1986 *Fractals in Physics* (Amsterdam: North-Holland)
- [6] Pynn R and Skjeltorp A (ed) 1985 *Scaling Phenomena in Disordered Systems* (New York: Plenum)
- [7] Stanley H E (ed) 1986 *Statistical Physics* (Amsterdam: North-Holland); 1986 *Physica* **140A** nos 1 and 2
- [8] Meakin P, Stanley H E, Coniglio A and Witten T A 1985 *Phys. Rev. A* **32** 2364; 1986 *Phys. Rev. A* **34** 3325
- [9] Meakin P 1987 *Phys. Rev. A* **35** 2234
- [10] Family F, Vicsek T and Taggett B 1986 *J. Phys. A: Math. Gen.* **19** L727
- [11] Hermann H J 1986 *Phys. Rep.* **136** 153
- [12] Paterson L 1984 *Phys. Rev. Lett.* **52** 1621
- [13] Vicsek T 1984 *Phys. Rev. Lett.* **53** 2281
- [14] Bensimon D, Kadanoff L P, Liang S, Shraiman B I and Tang C 1985 *Preprint*
- [15] Sahimi M and Yortsos Y C 1985 *Phys. Rev. A* **32** 3762
- [16] Sherwood J D and Nittmann J 1986 *J. Physique* **47** 15
- [17] King P R 1987 *J. Phys. A: Math. Gen.* **20** L529
- [18] Nagatani T 1987 *J. Phys. A: Math. Gen.* **20** L381
- [19] Nagatani T 1987 *Phys. Rev. A* in press
- [20] Nagatani T 1987 *J. Phys. A: Math. Gen.* **20** 6603
- [21] Ball R C 1986 *Statistical Physics* ed H E Stanley (Amsterdam: North-Holland) p 62
- [22] Turkevich L A and Scher H 1985 *Phys. Rev. Lett.* **55** 1026
- [23] Nagatani T 1987 *Phys. Rev. A* **35** 2765
- [24] Amtrano C, Coniglio A and di Liberto F 1986 *Phys. Rev. Lett.* **57** 1016
- [25] Hayakawa Y, Sato S and Matsushita M 1987 *Phys. Rev. A* **36** 795
- [26] Nagatani T 1987 *J. Phys. A: Math. Gen.* **20** L641